HÖLDER EQUICONTINUITY OF THE INTEGRATED DENSITY OF STATES AT WEAK DISORDER

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Abstract. Hölder continuity, $|N_\lambda(E) - N_\lambda(E')| \le C|E - E'|^\alpha$, with a constant C independent of the disorder strength λ is proved for the integrated density of states $N_\lambda(E)$ associated to a discrete random operator $H = H_o + \lambda V$ consisting of a translation invariant hopping matrix H_o and i.i.d. single site potentials V with an absolutely continuous distribution, under a regularity assumption for the hopping term.

1. Introduction

Random operators on $\ell^2(\mathbb{Z}^d)$ of the general form

$$H_\omega \;=\; H_o + \lambda V_\omega \;, \tag{1}$$

play a central role in the theory of disordered materials, where:

(1) $V_\omega\psi(x) = \omega(x)\psi(x)$ with $\omega(x)$, $x \in \mathbb{Z}^d$, independent identically distributed random variables whose common distribution is $\rho(\omega)\mathrm{d}\omega$ with ρ a bounded function. The coupling $\lambda \in \mathbb{R}$ is called the *disorder strength*.
(2) H_o is a bounded translation invariant operator, i.e., $[S_\xi, H_o] = 0$ for each translation $S_\xi\psi(x) = \psi(x - \xi)$, $\xi \in \mathbb{Z}^d$.

The *density of states measure* for an operator H_ω of the form eq. (1) is the (unique) Borel measure $\mathrm{d}N_\lambda(E)$ on the real line defined by

$$\int f(E)\mathrm{d}N_\lambda(E) \;=\; \lim_{L\to\infty} \frac{1}{\#\left\{x \in \mathbb{Z}^d : |x| < L\right\}} \sum_{x:|x|<L} \langle \delta_x, f(H_\omega)\delta_x\rangle \;,$$

and the *integrated density of states* $N_\lambda(E)$ is

$$N_\lambda(E) \;:=\; \int_{(-\infty,E)} \mathrm{d}N_\lambda(\varepsilon) \;.$$

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It is a well known consequence, e.g., ref [6], of the translation invariance of the distribution of H_ω that the density of states exists and equals

$$N_\lambda(E) \;=\; \int_\Omega \left\langle \delta_0, P_{(-\infty,E)}(H_\omega)\delta_0 \right\rangle \mathrm{d}\mathbb{P}(\omega) \;, \qquad \text{every } E \in \mathbb{R};$$

for $\mathbb{P}$ almost every ω, where $\mathbb{P}$ is the joint probability distribution for ω and $\Omega = \mathbb{R}^{\mathbb{Z}^d}$ is the probability space.

The density of states measure is an object of fundamental physical interest. For example, the free energy f per unit volume of a system of non-interacting identical Fermions, each governed by a Hamiltonian H_ω of the form eq. (1), is

$$f(\mu,\beta) \;=\; -\beta \int \ln(1 + \mathrm{e}^{-\beta(E-\mu)})\mathrm{d}N_\lambda(E) \;,$$

where β is the inverse temperature and μ is the chemical potential. Certain other thermodynamic quantities (density, heat capacity, etc.) of the system can also be expressed in terms of N_λ.

Our main result is equicontinuity of the family $\{N_\lambda(\cdot), \lambda > 0\}$ within a class of Hölder continuous functions, that is

$$N_\lambda(E+\delta) - N_\lambda(E-\delta) \;\le\; C_\alpha\, \delta^\alpha, \text{ for all } \lambda > 0 \;, \tag{2}$$

under appropriate hypotheses on H_o. The exponent $\alpha < 1$ depends on H_o as well as the probability density, with $\alpha = \frac{1}{2}$ at generic E for a large class of hopping terms if ρ is compactly supported.

A bound of the form eq. (2) for the integrated density of states associated to a continuum random Schrödinger operator is implicit in Theorem 1.1 of ref. [1], although uniformity in λ is not explicitly noted there. The tools of ref. [1] carry over easily to the discrete context to give an alternative proof of eq. (2). However the methods employed herein are in fact quite different from those of ref. [1], and may be interesting in and of themselves.

The main point of eq. (2) is the uniformity of the bound as $\lambda \to 0$, since the well known *Wegner estimate* [9], see also [7, Theorem 8.2],

$$\frac{\mathrm{d}N_\lambda(E)}{\mathrm{d}E} \;\le\; \frac{\|\rho\|_\infty}{\lambda} \;, \tag{3}$$

implies that $N_\lambda(E)$ is in fact Lipschitz continuous,

$$N_\lambda(E+\delta) - N_\lambda(E-\delta) \;\le\; \frac{\|\rho\|_\infty}{\lambda}\, 2\delta \;. \tag{4}$$

However, the Lipschitz constant $\|\rho\|_\infty / \lambda$ in eq. (4) diverges as $\lambda \to 0$. Such a singularity is inevitable for a bound which makes no reference to the hopping term, since $\mathrm{d}N_\lambda(E) = \lambda^{-1}\rho(E/\lambda)\mathrm{d}E$ for $H_o = 0$, as may easily be verified. However if the background itself has an absolutely

continuous density of states, the Wegner estimate is far from optimal at weak disorder.

The translation invariant operator H_o may be written as a superposition of translations,

$$H_o \ = \ \sum_{\xi} \check{\varepsilon}(\xi) S_\xi \ ,$$

where

$$\check{\varepsilon}(\xi) \ = \ \int_{T^d} \varepsilon(\mathbf{q}) \mathrm{e}^{-\mathrm{i}\xi\cdot\mathbf{q}} \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d} \ ,$$

is the inverse Fourier transform of a bounded real function ε on the torus $T^d = [0, 2\pi)^d$, called the symbol of H_o. For any bounded measurable function f,

$$f(H_o) \ = \ \sum_{\xi \in \mathbb{Z}^d} \left[\int_{T^d} f(\varepsilon(\mathbf{q})) \mathrm{e}^{-\mathrm{i}\xi\cdot\mathbf{q}} \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d} \right] S_\xi \ ,$$

from which it follows that the density of states $N_o(E)$ for H_o obeys

$$\int f(E) \mathrm{d}N_o(E) \ = \ \int_{T^d} f(\varepsilon(\mathbf{q})) \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d} \ .$$

In particular,

$$N_o(E) \ = \ \int_{\{\varepsilon(\mathbf{q}) < E\}} \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d} \ .$$

We define a *regular point* for ε to be a point $E \in \mathbb{R}$ at which

$$N_o(E+\delta) - N_o(E-\delta) \ \le \ \Gamma(E)\,\delta \ , \tag{5}$$

for some $\Gamma(E) < \infty$. In particular if ε is C^1 and $\nabla\varepsilon$ is non-zero on the level set $\{\varepsilon(\mathbf{q}) = E\}$, then E is a regular point. For example, with H_o the discrete Laplacian on $\ell^2(\mathbb{Z})$,

$$H_o \psi(x) \ = \ \psi(x+1) + \psi(x-1) \ ,$$

we have the symbol $\varepsilon(q) = 2\cos(q)$ and every $E \in (-2, 2)$ is a regular point. However at the band edges, $E = \pm 2$, the difference on the right hand side of eq. (5) is only $\mathcal{O}(\delta^{\frac{1}{2}})$, and these points are not regular points. We consider the behavior of $N_\lambda(E)$ at such "points of order α," here $\alpha = 1/2$, in Theorem 3 below.

Our main result involves the density of states of H_λ at a regular point:

Theorem 1. *Suppose* $\int |\omega|^q \rho(\omega) \mathrm{d}\omega < \infty$ *for some* $2 < q < \infty$ *or that* ρ *is compactly supported, in which case set* $q = \infty$*. If* E *is a regular point for* ε*, then there is* $C_q = C_q(\rho, \Gamma(E)) < \infty$ *such that*

$$N_\lambda(E+\delta) - N_\lambda(E-\delta) \ \le \ \Gamma(E)\,\delta \ + \ C_q\, \lambda^{\frac{1}{3}(1+\frac{2}{q})} \delta^{\frac{1}{3}(1-\frac{2}{q})} \tag{6}$$

for all $\lambda, \delta \geq 0$.

For very small δ, namely

$$\frac{\delta}{\lambda} \lesssim \lambda^{\frac{1}{3}(1+\frac{2}{q})} \delta^{\frac{1}{3}(1-\frac{2}{q})} ,$$

the Wegner bound eq. (3) is stronger than eq. (6).[1] Thus Theorem 1 is useful only for

$$\delta \gtrsim \lambda^{\frac{2q+1}{q+1}} .$$

Combining the Wegner estimate and Theorem 1 for these separate regions yields the following:

Corollary 2. *Under the hypotheses of Theorem 1, there is* $C_q < \infty$*, with* $C_q = C_q(\rho, \Gamma(E))$*, such that*

$$N_\lambda(E+\delta) - N_\lambda(E-\delta) \leq C_q \delta^{\frac{1}{2}(1-\frac{1}{2q+1})} \tag{7}$$

for all $\lambda, \delta \geq 0$.

Thus, the integrated density of states is Hölder equi-continuous of order $\frac{1}{2}$ as $\lambda \to 0$ (if ρ is compactly supported).

The starting point for our analysis of the density of states is a well known formula relating $\mathrm{d}N_\lambda$ to the resolvent of H_ω,

$$\frac{\mathrm{d}N_\lambda(E)}{\mathrm{d}E} = \lim_{\eta \downarrow 0} \frac{1}{\pi} \int_\Omega \mathrm{Im} \left\langle \delta_0, (H_\omega - E - \mathrm{i}\eta)^{-1} \delta_0 \right\rangle \mathrm{d}\mathbb{P}(\omega) .$$

The general idea of the proof is to express $\mathrm{Im} \langle \delta_0, (H_\omega - E - \mathrm{i}\eta)^{-1} \delta_0 \rangle$ using a finite resolvent expansion to second order

$$\begin{aligned} &(H_\omega - E - \mathrm{i}\eta)^{-1} \\ &= (H_o - E - \mathrm{i}\eta)^{-1} - \lambda (H_o - E - \mathrm{i}\eta)^{-1} V_\omega (H_o - E - \mathrm{i}\eta)^{-1} \\ &\quad + \lambda^2 (H_o - E - \mathrm{i}\eta)^{-1} V_\omega (H_\omega - E - \mathrm{i}\eta)^{-1} V_\omega (H_o - E - \mathrm{i}\eta)^{-1} , \end{aligned} \tag{8}$$

and to use the Wegner bound eq. (3) to estimate the last term, with the resulting factor of $1/\lambda$ controlled by the factor λ^2.

Here is a simplified version of the argument which works if E falls outside the spectrum of H_o and $\psi_E = (H_o - E)^{-1} \delta_0 \in \ell^1(\mathbb{Z}^d)$. The first

[1] We thank M. Disertori for this observation.

two terms of eq. (8) are bounded and self-adjoint when $\eta = 0$, so

$$\begin{aligned}
&\lim_{\eta \downarrow 0} \frac{1}{\pi} \int_\Omega \operatorname{Im} \left\langle \delta_0, (H_\omega - E - \mathrm{i}\eta)^{-1} \delta_0 \right\rangle \mathrm{d}\mathbb{P}(\omega) \\
&\quad = \lambda^2 \lim_{\eta \downarrow 0} \frac{1}{\pi} \int_\Omega \operatorname{Im} \left\langle \psi_E, V_\omega (H_\omega - E - \mathrm{i}\eta)^{-1} V_\omega \psi_E \right\rangle \mathrm{d}\mathbb{P}(\omega) \\
&\quad \leq \lambda^2 \lim_{\eta \downarrow 0} \sum_{x,y} |\psi_E(x)| \, |\psi_E(y)| \\
&\qquad\qquad \times \frac{\eta}{\pi} \int_\Omega \left| \omega(x)\omega(y) \left\langle \delta_x, \left((H_\omega - E)^2 + \eta^2 \right)^{-1} \delta_y \right\rangle \right| \mathrm{d}\mathbb{P}(\omega) \, .
\end{aligned}$$

If ρ is, say, compactly supported, then

$$\begin{aligned}
&\lim_{\eta \downarrow 0} \frac{\eta}{\pi} \int_\Omega \left| \omega(x)\omega(y) \left\langle \delta_x, \left((H_\omega - E)^2 + \eta^2 \right)^{-1} \delta_y \right\rangle \right| \mathrm{d}\mathbb{P}(\omega) \\
&\qquad\qquad \lesssim \lim_{\eta \downarrow 0} \frac{\eta}{\pi} \int_\Omega \left\langle \delta_x, \left((H_\omega - E)^2 + \eta^2 \right)^{-1} \delta_y \right\rangle \mathrm{d}\mathbb{P}(\omega) \lesssim \frac{1}{\lambda} \, ,
\end{aligned}$$

by the Wegner bound, and therefore

$$\frac{\mathrm{d}N_\lambda(E)}{\mathrm{d}E} \lesssim \lambda \left\| \psi_E \right\|_1^2 \, , \qquad \text{for } E \notin \sigma(H_o) \, . \tag{9}$$

We have used second order perturbation theory to "boot-strap" the Wegner estimate and obtain an estimate of lower order in λ. Unfortunately, as ρ was assumed compactly supported, E is not in the spectrum of H_λ for sufficiently small λ, and thus $\mathrm{d}N_\lambda(E)/\mathrm{d}E = 0$. So, in practice, eq. (9) is not a useful bound.

Nonetheless, in the cases covered by Theorem 1, H_λ can have spectrum in a neighborhood of E, even for small λ, since E may be in the interior of the spectrum of H_o. Although, the above argument does not go through, we shall exploit the translation invariance of the distribution of H_ω by introducing a Fourier transform on the Hilbert space of "random wave functions," complex valued functions $\Psi(x, \omega)$ of $(x, \omega) \in \ell^2(\mathbb{Z}^d) \times \Omega$ with

$$\sum_x \int_\Omega \left| \Psi(x, \omega) \right|^2 \mathrm{d}\mathbb{P}(\omega) < \infty \, .$$

Under this Fourier transform an integral $\int_\Omega$ of a matrix element of $f(H_\omega)$ is replaced by an integral $\int_{T^d}$ over the d-torus of a matrix element of $f(\widehat{H}_{\mathbf{k}})$, with $\widehat{H}_{\mathbf{k}}$ a certain family of operators on $L^2(\Omega)$ (see eq. (16)). Off the set $S_\varepsilon := \{\mathbf{k} \in T^d || \varepsilon(\mathbf{k}) - E| > \epsilon\}$ with $\epsilon >> \delta$, we are able to carry out an argument similar to that which led to eq. (9). To prove

Theorem 1, we shall directly estimate

$$N(E+\delta) - N(E-\delta) \ = \ \int_\Omega \langle \delta_0, P_\delta(H_\omega)\delta_0\rangle \, \mathrm{d}\mathbb{P}(\omega) \ ,$$

with P_δ the characteristic function of the interval $[E-\delta, E+\delta]$, because the integrand on the r.h.s. is bounded by 1. Since E is a regular point, the error in restricting to S_ε will be bounded by $\Gamma(E)\varepsilon$. Choosing ε optimally will lead to Theorem 1.

More generally, we say that E is a point of order α for ε, if there exists $\Gamma(E;\alpha)$ such that

$$N_o(E+\delta) - N_o(E-\delta) \ \le \ \Gamma(E;\alpha)\delta^\alpha \ .$$

If $E \notin \sigma(H_o)$, we say that E is a point of order ∞ and set $\Gamma(E;\infty) = 0$. For points of order α we have the following extension of Theorem 1.

Theorem 3. *Suppose* $\int |\omega|^q \rho(\omega)\mathrm{d}\omega < \infty$ *for some* $2 < q < \infty$ *or that* ρ *is compactly supported, in which case set* $q = \infty$*. If* E *is a point of order* $\alpha \le \infty$ *for* ε*, then there is* $C_{q,\alpha} = C_{q,\alpha}(\rho, \Gamma(E;\alpha)) < \infty$ *such that*

$$N_\lambda(E+\delta) - N_\lambda(E-\delta) \ \le \ \Gamma(E;\alpha)\delta^\alpha \ + \ C_{q,\alpha} \left[\lambda^{1+\frac{2}{q}}\delta^{1-\frac{2}{q}}\right]^{\frac{1}{1+\frac{2}{\alpha}}} \tag{10}$$

for all $\lambda, \delta \ge 0$.

When $\alpha = \infty$ and $q = \infty$, so $E \notin \sigma(H_o)$ and ρ is compactly supported, the result is technically true but uninteresting since $E \notin \sigma(H_\lambda)$ for small λ, as discussed above. However for $q < \infty$, we need not have that ρ is compactly supported, and $E \notin \sigma(H_o)$ may still be in the spectrum of H_λ for arbitrarily small λ. In this case, eq. (10) signifies that

$$N_\lambda(E+\delta) - N_\lambda(E-\delta) \ \le \ C_{q,\infty}\lambda^{1+\frac{2}{q}}\delta^{1-\frac{2}{q}} \ ,$$

which in fact improves on the Wegner bound for appropriate λ, δ.

As above, we may use the Wegner bound for δ very small to improve on eq. (10):

Corollary 4. *Under the hypotheses of Theorem 3, there is* $C_{q,\alpha} = C_{q,\alpha}(\rho, \Gamma(E;\alpha)) < \infty$ *such that*

$$N_\lambda(E+\delta) - N_\lambda(E-\delta) \ \le \ C_{q;\alpha}\delta^{\frac{\alpha}{\alpha+1}\left(1-\frac{1}{\frac{\alpha+1}{\alpha}q+1}\right)}$$

for all $\lambda, \delta \ge 0$.

The inspiration for these results is the (non-rigorous) renormalized perturbation theory for $\mathrm{d}N_\lambda$ which has appeared in the physics literature, e.g., ref. [8] and references therein. If $\int \omega\rho(\omega)\mathrm{d}\omega = 0$ and

$\int \omega^2 \rho(\omega) \mathrm{d}\omega = 1$, as can always be achieved by shifting the origin of energy and re-scaling λ, then the central result of that analysis is that

$$\frac{\mathrm{d}N_\lambda(E)}{\mathrm{d}E} \approx \frac{1}{\pi} \mathrm{Im} \left\langle \delta_0, \left(H_o - E - \lambda^2 \Gamma_\lambda(E)\right)^{-1} \delta_0 \right\rangle ,$$

where $\Gamma_\lambda(E)$, the so-called "self energy," satisfies $\mathrm{Im}\,\Gamma_\lambda(E) > 0$ with

$$\lim_{\lambda \to 0} \mathrm{Im}\,\Gamma_\lambda(E) \approx \lim_{\eta \to 0} \mathrm{Im} \left\langle \delta_0, (H_o - E - \mathrm{i}\eta)^{-1} \delta_0 \right\rangle = \pi \frac{\mathrm{d}N_0(E)}{\mathrm{d}E} .$$

Up to a point, the self-energy analysis may be followed rigorously. Specifically, one can show (see §2):

Proposition 1.1. *If* $\int \omega \rho(\omega) \mathrm{d}\omega = 0$ *and* $\int \omega^2 \rho(\omega) \mathrm{d}\omega = 1$*, then for each* $\lambda > 0$ *there is a map* Γ_λ *from* $\{\mathrm{Im} z > 0\}$ *to the translation invariant operators with non-negative imaginary part on* $\ell^2(\mathbb{Z}^2)$ *such that*

$$\int_\Omega (H_\omega - z)^{-1} \mathrm{d}\mathbb{P}(\omega) = \left(H_o - z - \lambda^2 \Gamma_\lambda(z)\right)^{-1} , \tag{11}$$

and for fixed $z \in \{\mathrm{Im} z > 0\}$

$$\lim_{\lambda \to 0} \langle \delta_x, \Gamma_\lambda(z) \delta_y \rangle = \left\langle \delta_0, (H_o - z)^{-1} \delta_0 \right\rangle \delta_{x,y} . \tag{12}$$

However there is *a priori* no uniformity in z for the convergence in eq. (12), so for fixed λ we may conclude nothing about

$$\lim_{\eta \downarrow 0} \left(H_o - E - \mathrm{i}\eta - \lambda^2 \Gamma_\lambda(E + \mathrm{i}\eta)\right)^{-1} .$$

Still, one is left feeling that Theorem 1 and Corollary 2 are not-optimal, and the "standard wisdom" is that something like the following is true.

Conjecture 5. *Let* ρ *have moments of all orders, i.e.,* $\int |\omega|^q \rho(\omega) < \infty$ *for all* $q \geq 1$*. Given* $E_o \in \mathbb{R}$*, if there is* $\delta > 0$ *such that on the set* $\{\mathbf{q} : |\varepsilon(\mathbf{q}) - E_o| < \delta\}$ *the symbol* ε *is* C^1 *with* $\nabla \varepsilon(\mathbf{q}) \neq 0$*, then there is* $C_\delta < \infty$ *such that*

$$\frac{\mathrm{d}N_\lambda(E)}{\mathrm{d}E} \leq C_\delta$$

for all $\lambda \in \mathbb{R}$ *and* $E \in [E_o - \frac{1}{2}\delta, E_o + \frac{1}{2}\delta]$.

Remark: The requirement that ρ have moments of all orders is simply the minimal requirement for the infinite perturbation series for $(H_o - z - \lambda V_\omega)^{-1}$ to have finite expectation at each order (for $\mathrm{Im} z > 0$). In fact, this may be superfluous, as suggested by the example of Cauchy

randomness, for which the density of states can be explicitly computed, see ref. [7]:

$$\mathrm{d}N_\lambda(E) \ = \ \frac{1}{\pi}\int_{T^d} \frac{\lambda}{(\varepsilon(\mathbf{q})-E)^2+\lambda^2}\frac{\mathrm{d}\mathbf{q}}{(2\pi)^d} \ , \quad \text{for } \rho(\omega) = \frac{1}{\pi}\frac{1}{1+\omega^2} \ ,$$

although $\int \rho(\omega)\,|\omega|^q = \infty$ for every $q \geq 1$.

2. Translation invariance, augmented space, and a Fourier transform

The joint probability measure $\mathbb{P}(\omega)$ for the random function $\omega : \mathbb{Z}^d \to \mathbb{R}$ is

$$\mathrm{d}\mathbb{P}(\omega) \ := \ \prod_{x\in\mathbb{Z}^d} \rho(\omega(x))\mathrm{d}\omega(x)$$

on the probability space $\Omega = \mathbb{R}^{\mathbb{Z}^d}$. Clearly, $\mathbb{P}(\omega)$ is invariant under the translations $\tau_\xi : \Omega \to \Omega$ defined by

$$\tau_\xi\omega(x) \ = \ \omega(x-\xi) \ .$$

In particular, since

$$S_\xi H_\omega S_\xi^\dagger \ = \ H_o + V_{\tau_\xi\omega} \ = \ H_{\tau_\xi\omega} \ , \tag{13}$$

H_ω and $S_\xi H_\omega S_\xi^\dagger$ are identically distributed for any $\xi \in \mathbb{Z}^d$,

To express this invariance in operator theoretic terms, we introduce the fibred action of H_ω on the Hilbert space $L^2(\Omega;\ell^2(\mathbb{Z}^d))$ – the space of "random wave functions" – namely,

$$\Psi(\omega) \ \mapsto \ H_\omega\Psi(\omega) \ .$$

We identify $L^2(\Omega;\ell^2(\mathbb{Z}^d))$ with $L^2(\Omega\times\mathbb{Z}^d)$ and denote the action of H_ω on the latter space by $\mathbf{H}$, so

$$[\mathbf{H}\Psi](\omega,x) \ = \ \sum_\xi \check{\varepsilon}(\xi)\Psi(\omega,x-\xi) \ + \ \lambda\omega(x)\Psi(\omega,x) \ .$$

The following elementary identity relates $\int_\Omega f(H_\omega)\mathrm{d}\mathbb{P}(\omega)$ to $f(\mathbf{H})$, for any bounded measurable function f,

$$\int_\Omega \mathrm{d}\mathbb{P}(\omega)\,\langle\delta_x, f(H_\omega)\delta_y\rangle \ = \ \left\langle \mathbb{E}^\dagger\delta_x, f(\mathbf{H})\mathbb{E}^\dagger\delta_y\right\rangle \ , \tag{14}$$

where $\mathbb{E}^\dagger$ is the adjoint of the linear expectation map $\mathbb{E} : L^2(\Omega\times\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ defined by

$$[\mathbb{E}\Psi](x) \ = \ \int_\Omega \Psi(\omega,x)\mathrm{d}\mathbb{P}(\omega) \ .$$

Note that $\mathbb{E}^\dagger$ is an isometry from $\ell^2(\mathbb{Z}^d)$ onto the subspace of functions independent of ω – "non-random functions."

The general fact that averages of certain quantities depending on H_ω can be represented as matrix elements of $\mathbf{H}$ is known, and is sometimes called the "augmented space representation" (e.g., ref. [4, 5, 3]) where "augmented space" refers to the Hilbert space $L^2(\Omega \times \mathbb{Z}^d)$. There are "augmented space" formulae other than eq. (14), such as

$$\int_\Omega \mathrm{d}\mathbb{P}(\omega)\omega(x)\omega(y)\left\langle \delta_x, f(H_\omega)\delta_y\right\rangle \;=\; \left\langle \mathbb{E}^\dagger\delta_x, \mathbf{V}f(\mathbf{H})\mathbf{V}\mathbb{E}^\dagger\delta_y\right\rangle \;, \qquad (15)$$

and

$$\int_\Omega \mathrm{d}\mathbb{P}(\omega)\left\langle \delta_x, f(H_\omega)\delta_0\right\rangle\left\langle \delta_0, g(H_\omega)\delta_y\right\rangle \;=\; \left\langle \mathbb{E}^\dagger\delta_x, f(\mathbf{H})P_0 g(\mathbf{H})\mathbb{E}^\dagger\delta_y\right\rangle \;,$$

where P_0 denotes the projection $P_0\Psi(\omega,x) \;=\; \Psi(\omega,0)$ if $x=0$ and 0 otherwise. The first of these (eq. (15)) will play a roll in the proof of Theorem 1.

There are two natural groups of unitary translations on $L^2(\Omega\times\mathbb{Z}^d)$:

$$S_\xi\Psi(\omega,x) = \Psi(\omega, x-\xi)\;,$$

and

$$T_\xi\Psi(\omega,x) \;=\; \Psi(\tau_{-\xi}\omega, x)\;.$$

Note that these groups commute: $[S_\xi, T_{\xi'}] = 0$ for every $\xi,\xi' \in \mathbb{Z}^d$. A key observation is that the distributional invariance of H_ω, eq. (13), results in the *invariance* of $\mathbf{H}$ under the combined translations $T_\xi S_\xi = S_\xi T_\xi$:

$$S_\xi T_\xi \mathbf{H} T_\xi^\dagger S_\xi^\dagger \;=\; \mathbf{H}\;.$$

In fact, let us define

$$\mathbf{H}_o \;=\; \sum_\xi \check{\varepsilon}(\xi)S_\xi\;, \qquad \mathbf{V}\Psi(\omega,x) \;=\; \omega(x)\Psi(\omega,x)\;.$$

Then $\mathbf{H} = \mathbf{H}_o + \lambda\mathbf{V}$ where $\mathbf{H}_o$ commutes with S_ξ and T_ξ while for $\mathbf{V}$ we have

$$\mathbf{V}S_\xi \;=\; T_{-\xi}\mathbf{V}\;.$$

To exploit this translation invariance of $\mathbf{H}$, we define a Fourier transform which diagonalizes the translations $S_\xi T_\xi$ (and therefore partially diagonalizes $\mathbf{H}$). The result is a unitary map $\mathcal{F} : L^2(\Omega\times\mathbb{Z}^d) \to L^2(\Omega\times T^d)$, with T^d the d-torus $[0,2\pi)^d$. Let us define $\mathcal{F}$ first on functions having finite support in $\mathbb{Z}^d$ by

$$\mathcal{F}\Psi(\omega,\mathbf{k}) \;=\; \sum_\xi \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\xi}\Psi(-\xi, \tau_{-\xi}\omega)\;.$$

It is easy to verify, using well known properties of the usual Fourier series mapping $\ell^2(\mathbb{Z}^d) \to L^2(T^d)$, that $\mathcal{F}$ extends to a unitary isomorphism $L^2(\Omega \times \mathbb{Z}^d) \to L^2(\Omega \times T^d)$, i.e. that $\mathcal{F}\mathcal{F}^\dagger = \mathbf{1}$ and $\mathcal{F}^\dagger\mathcal{F} = \mathbf{1}$ where $\mathcal{F}^\dagger$ is the adjoint map

$$\mathcal{F}^\dagger\widehat{\Psi}(\omega, x) \ = \ \int_{T^d} \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot x}\widehat{\Psi}(\tau_{-x}\omega, \mathbf{k})\frac{\mathrm{d}\mathbf{k}}{(2\pi)^d} \ .$$

Another way of looking at $\mathcal{F}$ is to define for each $\mathbf{k} \in T^d$ an operator $\mathcal{F}_{\mathbf{k}} : L^2(\Omega \times \mathbb{Z}^d) \to L^2(\Omega)$ by

$$\mathcal{F}_{\mathbf{k}}\Psi \ = \ \lim_{L\to\infty} \sum_{|\xi|<L} \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\xi}\mathcal{J} S_\xi T_\xi \Psi \ ,$$

where $\mathcal{J}$ is the evaluation map $\mathcal{J}\Psi(\omega) \ = \ \Psi(\omega, 0)$. The maps $\mathcal{F}_{\mathbf{k}}$ are *not* bounded, but are densely defined with $\mathcal{F}_{\mathbf{k}}\Psi \in L^2(\Omega)$ for almost every $\mathbf{k}$, and

$$\mathcal{F}\Psi(\omega, \mathbf{k}) \ = \ \mathcal{F}_{\mathbf{k}}\Psi(\omega) \quad \text{a.e. } \omega, \mathbf{k}.$$

If we look at $L^2(\Omega \times T^d)$ as the direct integral $\int^\oplus \mathrm{d}\mathbf{k} L^2(\Omega)$, then

$$\mathcal{F} = \int^\oplus \mathrm{d}\mathbf{k}\mathcal{F}_{\mathbf{k}} \ .$$

This Fourier transform diagonalizes the combined translation $S_\xi T_\xi$,

$$\mathcal{F}_{\mathbf{k}} S_\xi T_\xi \ = \ \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\xi}\mathcal{F}_{\mathbf{k}} \ ,$$

as follows from the following identities for S and T,

$$\mathcal{F}_{\mathbf{k}} T_\xi \ = \ T_\xi \mathcal{F}_{\mathbf{k}} \ , \quad \mathcal{F}_{\mathbf{k}} S_\xi \ = \ \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\xi} T_{-\xi}\mathcal{F}_{\mathbf{k}} \ ,$$

where, on the right hand sides, T_ξ denotes the operator $T_\xi\psi(\omega) \ = \ \psi(\tau_{-\xi}\omega)$ on $L^2(\Omega)$. Furthermore, explicit computation shows that

$$\mathcal{F}_{\mathbf{k}}\mathbf{V} \ = \ \omega(0)\mathcal{F}_{\mathbf{k}} \ ,$$

where $\omega(0)$ denotes the operator of multiplication by the random variable $\omega(0)$, $\psi(\omega) \ \mapsto \ \omega(0)\psi(\omega)$. Putting this all together yields

Proposition 2.1. *Under the natural identification of $L^2(\Omega, T^d)$ with the direct integral $\int^\oplus \mathrm{d}\mathbf{k} L^2(\Omega)$, the operator $\widehat{\mathbf{H}} = \mathcal{F}\mathbf{H}\mathcal{F}^\dagger$ is partially diagonalized, $\widehat{\mathbf{H}} = \int^\oplus \widehat{H}_{\mathbf{k}}$, with $\widehat{H}_{\mathbf{k}}$ operators on $L^2(\Omega)$ given by the following formula*

$$\widehat{H}_{\mathbf{k}} \ = \ \sum_\xi \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\xi}\check{\varepsilon}(-\xi)T_\xi \ + \ \lambda\omega(0) \ .$$

Let us introduce for each $\mathbf{k} \in T^d$,

$$\widehat{H}^o_{\mathbf{k}} \;:=\; \sum_\xi \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\xi}\check{\varepsilon}(-\xi)T_\xi \;=\; \sum_\xi \left[\int_{T^d} \varepsilon(\mathbf{q}+\mathbf{k})\mathrm{e}^{\mathrm{i}\xi\cdot\mathbf{q}}\,\frac{\mathrm{d}\mathbf{q}}{(2\pi)^d}\right] T_\xi \;,$$

so $\widehat{H}_{\mathbf{k}} = \widehat{H}^o_{\mathbf{k}} + \lambda\omega(0)$. Note that

$$\widehat{H}^o_{\mathbf{k}}\chi_\Omega \;=\; \varepsilon(\mathbf{k})\chi_\Omega \;,$$

where $\chi_\Omega(\omega) = 1$ for every $\omega \in \Omega$. That is, χ_Ω is an eigenvector for $H^o_{\mathbf{k}}$.[2]

Applying the Fourier transform $\mathcal{F}$ to the right hand side of the "augmented space" formula eq. (14) we obtain the following beautiful identity, central to this work:

$$\int_\Omega \mathrm{d}\mathbb{P}(\omega)\,\langle\delta_x, f(H_\omega)\delta_y\rangle \;=\; \int_{T^d}\frac{\mathrm{d}\mathbf{k}}{(2\pi)^d}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(x-y)}\left\langle\chi_\Omega, f(\widehat{H}_{\mathbf{k}})\chi_\Omega\right\rangle \;. \quad (16)$$

Similarly, we obtain

$$\int_\Omega \mathrm{d}\mathbb{P}(\omega)\omega(x)\omega(y)\,\langle\delta_x, f(H_\omega)\delta_y\rangle$$
$$= \int_{T^d}\frac{\mathrm{d}\mathbf{k}}{(2\pi)^d}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(x-y)}\left\langle\omega(0)\chi_\Omega, f(\widehat{H}_{\mathbf{k}})\omega(0)\chi_\Omega\right\rangle \quad (17)$$

from eq. (15). Related formulae have been used, for example, to derive the Aubry duality between strong and weak disorder for the almost Mathieu equation, see ref. [2] and references therein.

As a first application of eq. (16), let us prove the existence of the self energy (Prop. 1.1) starting from the identity

$$\int_\Omega \mathrm{d}\mathbb{P}(\omega)\left\langle\delta_0, (H_\omega - z)^{-1}\,\delta_0\right\rangle \;=\; \int_{T^d}\frac{\mathrm{d}\mathbf{k}}{(2\pi)^d}\left\langle\chi_\Omega, \left(\widehat{H}_{\mathbf{k}} - z\right)^{-1}\chi_\Omega\right\rangle \;.$$

Proof of Prop. 1.1. Since χ_Ω is an eigenvector of $\widehat{H}^o_{\mathbf{k}}$ and

$$\langle\chi_\Omega, \omega(0)\chi_\Omega\rangle \;=\; \int \omega\rho(\omega)\mathrm{d}\omega \;=\; 0\;,$$

the Feschbach mapping implies

$$\left\langle\chi_\Omega, \left(\widehat{H}_{\mathbf{k}} - z\right)^{-1}\chi_\Omega\right\rangle \;=\; \left(\varepsilon(k) - z - \lambda^2\Gamma_\lambda(z;\mathbf{k})\right)^{-1}\;, \quad (18)$$

[2]In fact, if ε is almost everywhere non-constant (so H_o has no eigenvalues) then $\varepsilon(k)$ is the *unique* eigenvalue for $\widehat{H}^o_{\mathbf{k}}$ and the remaining spectrum of $\widehat{H}^o_{\mathbf{k}}$ is infinitely degenerate absolutely continuous spectrum. One way to see this is to let $\phi_n(v)$ be the orthonormal polynomials with respect the weight $\rho(v)$, and look at the action of $\widehat{H}^o_{\mathbf{k}}$ on the basis for $L^2(\Omega)$ consisting of products of the form $\prod_{x\in\mathbb{Z}^d}\phi_{n(x)}(\omega(x))$ with only finitely many $n(x) \neq 0$.

with

$$\Gamma_\lambda(z;\mathbf{k}) \;=\; \left\langle \omega(0)\chi_\Omega, \left(P^\perp \widehat{H}_{\mathbf{k}} P^\perp - z\right)^{-1} \omega(0)\chi_\Omega \right\rangle \;,$$

where $P^\perp$ denotes the projection onto the orthogonal complement of χ_Ω in $L^2(\Omega)$.

Let the self energy $\Gamma_\lambda(z)$ be the translation invariant operator with symbol $\Gamma_\lambda(z;\mathbf{k})$, i.e.,

$$\langle \delta_x, \Gamma_\lambda(z)\delta_y\rangle \;=\; \int_{T^d} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(x-y)} \Gamma_\lambda(z;\mathbf{k}) \frac{\mathrm{d}\mathbf{k}}{(2\pi)^d} \;.$$

Clearly $\Gamma_\lambda(z)$ is bounded with non-negative imaginary part. Furthermore by eq. (16) and eq. (18), the identity eq. (11) holds, namely

$$\int_\Omega (H_\omega - z)^{-1} \mathrm{d}\mathbb{P}(\omega) \;=\; \left(H_o - z - \lambda^2 \Gamma_\lambda(z)\right)^{-1} \;.$$

It is clear that

$$\lim_{\lambda\to 0} \Gamma_\lambda(z;\mathbf{k}) \;=\; \left\langle \omega(0)\chi_\Omega, \left(\widehat{H}^o_{\mathbf{k}} - z\right)^{-1} \omega(0)\chi_\Omega \right\rangle \;,$$

from which eq. (12) follows easily. □

3. Proofs

We first prove Theorem 1 and then describe modifications of the proof which imply Theorem 3.

3.1. **Proof of Theorem 1.** Fix a regular point E for ε, and for each $\delta > 0$ let

$$\begin{aligned} f_\delta(t) \;&=\; \frac{1}{2}\left(\chi_{(E-\delta,E+\delta)}(t) + \chi_{[E-\delta,E+\delta]}(t)\right) \\ &=\; \begin{cases} 1\,, & t\,,\in (E-\delta, E+\delta)\,, \\ \frac{1}{2} & t = E\pm\delta\,, \\ 0 & t \notin [E-\delta, E+\delta]\,. \end{cases} \end{aligned}$$

Since $N_\lambda(E)$ is continuous (see eq. (4)),

$$N_\lambda(E+\delta) - N_\lambda(E-\delta) \;=\; \int_\Omega \langle \delta_0, f_\delta(H_\omega)\delta_0\rangle \,\mathrm{d}\mathbb{P}(\omega) \;.$$

Thus, in light of eq. (16), our task is to show that

$$\int_{T^d} \left\langle \chi_\Omega, f_\delta(\widehat{H}_{\mathbf{k}})\chi_\Omega \right\rangle \frac{\mathrm{d}\mathbf{k}}{(2\pi)^d} \;\le\; \Gamma(E)\,\delta \;+\; C_q\,\lambda^{\frac{1}{3}(1+\frac{2}{q})} \delta^{\frac{1}{3}(1-\frac{2}{q})} \;, \qquad (19)$$

with a constant C_q independent of δ and λ. Note that for each $\mathbf{k} \in T^d$

$$\left| \left\langle \chi_\Omega, f_\delta(\widehat{H}_{\mathbf{k}})\chi_\Omega \right\rangle \right| \;\le\; 1 \;,$$

so we can afford to neglect a set of Lebesgue measure $\lambda^{\frac{1}{3}(1+\frac{2}{q})}\delta^{\frac{1}{3}(1-\frac{2}{q})}$ on the l.h.s. of eq. (19).

Consider $\mathbf{k} \in T^d$ with $|\varepsilon(\mathbf{k}) - E| > \delta$. Then

$$f_\delta(\widehat{H}^o_{\mathbf{k}})\chi_\Omega \;=\; f_\delta(\varepsilon(\mathbf{k}))\chi_\Omega \;=\; 0\;.$$

Thus

$$\begin{aligned}
\left\langle \chi_\Omega, f_\delta(\widehat{H}_{\mathbf{k}})\chi_\Omega \right\rangle \;&=\; \left\langle \chi_\Omega, \left(f_\delta(\widehat{H}_{\mathbf{k}}) - f_\delta(\widehat{H}^o_{\mathbf{k}}) \right) \chi_\Omega \right\rangle \\
&=\; \lim_{\eta\to 0} \frac{1}{\pi} \int_{E-\delta}^{E+\delta} \mathrm{Im} \left\langle \chi_\Omega, \left(\frac{1}{\widehat{H}_{\mathbf{k}} - t - \mathrm{i}\eta} - \frac{1}{\widehat{H}^o_{\mathbf{k}} - t - \mathrm{i}\eta} \right) \chi_\Omega \right\rangle \mathrm{d}t \\
&=\; \lambda \lim_{\eta\to 0} \frac{1}{\pi} \int_{E-\delta}^{E+\delta} \mathrm{Im} \frac{1}{t + \mathrm{i}\eta - \varepsilon(\mathbf{k})} \left\langle \chi_\Omega, \frac{1}{\widehat{H}_{\mathbf{k}} - t - \mathrm{i}\eta} \omega(0)\chi_\Omega \right\rangle \mathrm{d}t \\
&=\; \lambda \left\langle \chi_\Omega, \frac{1}{\widehat{H}_{\mathbf{k}} - \varepsilon(\mathbf{k})} f_\delta(\widehat{H}_{\mathbf{k}}) \omega(0)\chi_\Omega \right\rangle \;,
\end{aligned} \tag{20}$$

since $(t - \varepsilon(\mathbf{k}))^{-1}$ is continuous for $t \in [E-\delta, E+\delta]$. Using again that $f_\delta(\widehat{H}^o_{\mathbf{k}})\chi_\Omega = 0$, we find that the final term of eq. (20) equals

$$\begin{aligned}
&= \left\langle \left[\frac{1}{\widehat{H}_{\mathbf{k}} - \varepsilon(\mathbf{k})} f_\delta(\widehat{H}_{\mathbf{k}}) - \frac{1}{\widehat{H}^o_{\mathbf{k}} - \varepsilon(\mathbf{k})} f_\delta(\widehat{H}^o_{\mathbf{k}}) \right] \chi_\Omega, \omega(0)\chi_\Omega \right\rangle \\
&= \lambda \lim_{\eta\to 0} \frac{1}{\pi} \int_{E-\delta}^{E+\delta} \frac{1}{t - \varepsilon(\mathbf{k})} \mathrm{Im} \frac{1}{t + \mathrm{i}\eta - \varepsilon(\mathbf{k})} \\
&\qquad\qquad \times \left\langle \frac{1}{\widehat{H}_{\mathbf{k}} - t - \mathrm{i}\eta} \omega(0)\chi_\Omega, \omega(0)\chi_\Omega \right\rangle \mathrm{d}t \\
&=\; \lambda \left\langle \omega(0)\chi_\Omega, \frac{f_\delta(\widehat{H}_{\mathbf{k}})}{(\widehat{H}_{\mathbf{k}} - \varepsilon(\mathbf{k}))^2} \omega(0)\chi_\Omega \right\rangle \;.
\end{aligned} \tag{21}$$

Putting eqs. (20) and (21) together yields

$$\begin{aligned}
\left\langle \chi_\Omega, f_\delta(\widehat{H}_{\mathbf{k}})\chi_\Omega \right\rangle \;&=\; \lambda^2 \left\langle \omega(0)\chi_\Omega, \frac{f_\delta(\widehat{H}_{\mathbf{k}})}{(\widehat{H}_{\mathbf{k}} - \varepsilon(\mathbf{k}))^2} \omega(0)\chi_\Omega \right\rangle \\
&\le\; \lambda^2 \frac{1}{(|\varepsilon(\mathbf{k}) - E| - \delta)^2} \left\langle \omega(0)\chi_\Omega, f_\delta(\widehat{H}_{\mathbf{k}}) \omega(0)\chi_\Omega \right\rangle \;.
\end{aligned}$$

Thus, for any $\epsilon > \delta$,

$$\int_{\{|\varepsilon(\mathbf{k})-E|>\epsilon\}} \left\langle \chi_\Omega, f_\delta(\widehat{H}_{\mathbf{k}})\chi_\Omega \right\rangle$$
$$\leq \lambda^2 \frac{1}{(\epsilon-\delta)^2} \int_{T^d} \left\langle \omega(0)\chi_\Omega, f_\delta(\widehat{H}_{\mathbf{k}})\omega(0)\chi_\Omega \right\rangle$$
$$= \lambda^2 \frac{1}{(\epsilon-\delta)^2} \int_\Omega \omega(0)^2 \langle \delta_0, f_\delta(H_\omega)\delta_0\rangle \, \mathrm{d}\mathbb{P}(\omega) \; ,$$

where in the last equality we have inverted the Fourier transform, using eq. (17). We may estimate the right hand side with Hölder's inequality and the Wegner estimate:

$$\int_\Omega \omega(0)^2 \langle \delta_0, f_\delta(H_\omega)\delta_0\rangle \, \mathrm{d}\mathbb{P}(\omega)$$
$$\leq \|\omega(0)\|_q^2 \left(\int_\Omega \langle \delta_0, f_\delta(H_\omega)\delta_0\rangle \, \mathrm{d}\mathbb{P}(\omega)\right)^{1-\frac{2}{q}}$$
$$\leq \|\omega(0)\|_q^2 \left(\frac{\|\rho\|_\infty}{\lambda} 2\delta \right)^{1-\frac{2}{q}} \quad ,$$

since $\langle \delta_0, f_\delta(H_\omega)\delta_0\rangle^p \leq \langle \delta_0, f_\delta(H_\omega)\delta_0\rangle$ for $p > 1$ (because $\langle \delta_0, f_\delta(H_\omega)\delta_0\rangle \leq 1$). Here $\|\omega(0)\|_q^q = \int \omega(0)^q \mathrm{d}\mathbb{P}(\omega)$ for $q < \infty$ and $\|\omega(0)\|_\infty =$ ess-sup$_\omega |\omega(0)|$.

Therefore

$$\int_{T^d} \left\langle \chi_\Omega, f_\delta(\widehat{H}_{\mathbf{k}})\chi_\Omega \right\rangle \;\leq\; \Gamma(E)\epsilon \;+\; \lambda^2 \frac{1}{(\epsilon-\delta)^2} \|\omega(0)\|_q^2 \left(\frac{\|\rho\|_\infty}{\lambda} 2\delta \right)^{1-\frac{2}{q}} \; , \tag{22}$$

where the first term on the right hand side is an upper bound for

$$\int_{\{|\varepsilon(\mathbf{k})-E|\leq\epsilon\}} \left\langle \chi_\Omega, f_\delta(\widehat{H}_{\mathbf{k}})\chi_\Omega \right\rangle \frac{\mathrm{d}\mathbf{k}}{(2\pi)^d} \;\leq\; \int_{\{|\varepsilon(\mathbf{k})-E|\leq\epsilon\}} \frac{\mathrm{d}\mathbf{k}}{(2\pi)^d} \; .$$

Upon optimizing over $\epsilon \in (\delta, \infty)$, this implies

$$\int_\Omega \langle \delta_0, f_\delta(H_\omega)\delta_o\rangle \;\leq\; \Gamma(E)\delta \;+\; C_{\rho,q,\Gamma}\, \lambda^{\frac{1}{3}(1+\frac{2}{q})} \delta^{\frac{1}{3}(1-\frac{2}{q})} \; ,$$

which completes the proof of Theorem 1. □

3.2. **Proof of Theorem 3.** If instead of being a regular point, E is a point of order α then the proof goes through up to eq. (22), in place of which we have

$$\int_{T^d} \left\langle \chi_\Omega, f_\delta(\widehat{H}_{\mathbf{k}})\chi_\Omega \right\rangle \;\leq\; \Gamma(E;\alpha)\epsilon^\alpha + \lambda^2 \frac{1}{(\epsilon-\delta)^2} \|\omega(0)\|_q^2 \left(\frac{\|\rho\|_\infty}{\lambda} \delta \right)^{1-\frac{2}{q}} \; .$$

Setting $\varepsilon = \delta + \lambda^\gamma \delta^\beta$ and choosing γ, β such that the two terms are of the same order yields

$$\gamma \;=\; \frac{1}{2+\alpha}\left(1 + \frac{2}{q}\right) , \qquad \beta \;=\; \frac{1}{2+\alpha}\left(1 - \frac{2}{q}\right) ,$$

which implies

$$\int_\Omega \langle \delta_0, f_\delta(H_\omega)\delta_o\rangle \;\le\; \Gamma(E;\alpha)\delta^\alpha \;+\; C_q\, \lambda^{\frac{\alpha}{2+\alpha}\left(1+\frac{2}{q}\right)}\; \delta^{\frac{\alpha}{2+\alpha}\left(1-\frac{2}{q}\right)} ,$$

completing the proof. □

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